Holonomy on *D*-branes

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Abstract

This paper shows how to construct anomaly free world sheet actions in string theory with *D*-branes. Our method is to use Deligne cohomology and bundle gerbe theory to define geometric objects which are naturally associated to *D*-branes and connections on them. The holonomy of these connections can be used to cancel global anomalies in the world sheet action.
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1. Introduction

It has been noted by a number of authors particularly Freed–Witten and Kapustin [6,12] that the *B*-field, in *D*-brane theory, defines a Deligne cohomology class and this interpretation has been used to show how anomaly cancellation occurs in the world sheet action.

The mathematical formalism underlying these observations starts with a space-time manifold M with a submanifold $Q \subset M$, the *D*-brane, and a good open cover $\mathcal{U} = \{U_\alpha\}_{a \in I}$ of M (recall that this means that every finite intersection of elements in $\mathcal{U}$ is contractible). The *B*-field is a collection of smooth de Rham two-forms $\{B_\alpha\}_{a \in I}$ with B_α defined on U_α and

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satisfying $dB_\alpha = dB_\beta$ on $U_\alpha \cap U_\beta$ for all $\alpha, \beta \in I$. Thus $\{dB_\alpha\}_{\alpha \in I}$ defines a three-form H on M. For full treatments, see [5,18,23,24].

The B-field has various mathematical interpretations which depend on associated topological and geometric structures. These interpretations include: a Cech representative for Deligne cohomology, a differential character, a connection and curving on a gerbe, a connection and curving on a bundle gerbe, a connection on a BS^1 bundle [7] or some differential geometric structure on a $PU(H)$ bundle. It is not clear to us if physics can distinguish between these different mathematical realisations of a B-field. In this paper we focus on the differential character or Deligne class which is the minimal geometric datum necessary to build world sheet actions.

In the simplest case the B-field restricts on the D-brane Q to the Stiefel–Whitney class of the normal bundle to Q. Then world sheet anomaly cancellation, or equivalently, the construction of world sheet actions was investigated in [6]. In this paper we show that for this case the differential character viewpoint alone suffices. This refines the results of [6] in that it eliminates any dependence of the action on choices such as open covers and makes explicit some other necessary but subtle choices (2.2) which affect the definition of the action.

In order to build a world sheet action in the more difficult situation where there is a general torsion B-field on Q (that is $H = 0$ on Q) we need to introduce bundle gerbes and bundle gerbe modules. These provide an alternative to the Azumaya algebra modules of [12]. Our approach provides a refinement of the conclusions of [12] in making explicit the extent of dependence on choices made in the construction.

Finally, bundle gerbe modules with infinite dimensional fibre are needed for world sheet actions in the presence of a non-torsion B-field (i.e. H is non-zero on Q, see [1,11]). This case has not been successfully treated previously. We propose here a way to produce an anomaly free world sheet action (this is the main result in the paper).

The paper is organised as follows. Section 2 contains an overview of our constructions without proofs. Proofs of the assertions in Section 2 are presented in the remaining sections.

We review some results on Deligne cohomology, its holonomy and the differential character in Section 3. Included here is a discussion of the notion of transgressing a Deligne two class on a manifold to a Deligne one class on the loop space of the manifold although our account emphasises the transgression of the differential character. This is sufficient to handle the situation considered in [6].

For the more difficult case of anomaly cancellation in the presence of general torsion B-fields we need more mathematical structure. This is because the world sheet action is a priori a section of a non-trivial line bundle. We use bundle gerbes and bundle gerbe modules to introduce new line bundles which can be tensored with the original line bundle and trivialise it.

Section 4.1 reviews relevant aspects of our earlier paper [2] where we used bundle gerbes to give a geometric approach to twisted K-theory. Here we explain the geometry of bundle gerbes and their relation to Deligne cohomology to connect up with the discussion of Section 3. Understanding how the action depends on choices made in its definition requires us to study gauge transformations of a bundle gerbe. These generalise the familiar idea of a gauge transformation on a line bundle. Then in Section 4.11 the holonomy of a connection on a bundle gerbe module is introduced motivated by the analogous construction in [12] for

Azumaya algebra modules. For torsion B-fields, as in [12], the trace of this bundle gerbe holonomy is a section of a line bundle over the loop space of Q and we tensor this on to our original world sheet action in order to convert it from a section of a non-trivial line bundle to a function.

Our main observation is that a modification of this construction may be used (Section 4.11) to handle the case of non-torsion B-field.

Finally, in Section 5, we round out our account by explaining the relationship with the approach in [12]. The point is that in the case of torsion B-field there is a groupoid C^*-algebra with spectrum M which acts on bundle gerbe modules over M. This groupoid C^*-algebra is continuous trace and hence has a Dixmier–Douady class. It is then Morita equivalent to any Azumaya algebra with spectrum M having the same Dixmier–Douady class. Azumaya algebras and their modules are used in [12] to construct twisted K-theory. It follows then that the K-theory of the groupoid algebra and the Azumaya algebra are the same whenever their Dixmier–Douady classes are equal and both give the twisted K-theory of M. However we do not take the C^*-algebra approach further because we do not know how to make it work in the non-torsion case.

2. Action building

In this section we list some basic facts about Deligne cohomology and show how they can be used to generate anomaly free world sheet actions. In the subsequent sections we give the mathematical background necessary to establish these facts.

Let M be a manifold with a submanifold Q. Let Σ be a Riemann surface with a single boundary component which is identified with the circle S^1. Denote by $\Sigma(M)$ the space of all maps of Σ into M and by $L(M)$ the space of all maps of the circle S^1 into M. By restricting a map of Σ into M to the boundary circle we obtain a map of the circle into M. This defines a map we call $\partial : \Sigma(M) \to L(M)$. We will be particularly interested in the subset of maps of Σ into M which map the boundary circle into the submanifold Q. We denote these by $\Sigma_Q(M)$. There is a commuting diagram

$$
\begin{array}{ccc}
\Sigma(M) & \xrightarrow{\partial} & L(M) \\
\cup & & \cup \\
\Sigma_Q(M) & \xrightarrow{\partial} & L(Q)
\end{array}
\tag{2.1}
$$

World sheet actions are functions on $\Sigma_Q(M)$. The world sheet actions that we are interested in will arise from sections of line bundles $L \to \Sigma_Q(M)$ constructed from geometric objects on Q and M. The primary geometric object we are interested in is the Deligne class which is a geometric interpretation of the B-field.

Let us review some basic facts about Deligne cohomology. On a manifold X there is the group $H^p(X, \mathcal{D}^p)$ of Deligne p classes. For now we need only a few results about this.

2.1. Properties of Deligne classes

First there is a homomorphism (see Section 3.2)

$$c : H^p(X, \mathcal{D}^p) \to H^{p+1}(X, \mathbb{Z}).$$

If ξ is a Deligne p class we call $c(\xi)$ its (characteristic) class.

There is also a homomorphism (Section 3.2)

$$\iota : \Omega^p(X) \to H^p(X, \mathcal{D}^p),$$

which sends a p form ρ to a Deligne class $\iota(\rho)$ and we have $c(\iota(\rho)) = 0$ for any p form ρ. The kernel of ι is $\Omega_c^p(X)$ the space of all closed p-forms whose integral over any closed submanifold is $2\pi i$ times an integer. This discussion is summarised by the exact sequence of groups (3.8)

$$0 \to \Omega_c^p(X) \to \Omega^p(X) \xrightarrow{\iota} H^p(X, \mathcal{D}^p) \xrightarrow{c} H^{p+1}(X, \mathbb{Z}) \to 0.$$

There is also a map

$$H^p(X, \mathcal{D}^p) \to \Omega^{p+1}(X),$$

which associates to a Deligne class ξ its *curvature* F_ξ which is a closed form. The de Rham class of F_ξ is the image of $2\pi i c(\xi)$ in real cohomology.

If $\gamma : \Sigma \to X$ is a map of a p-dimensional manifold Σ into X and $\xi \in H^p(X, \mathcal{D}^p)$ is a Deligne class there is a *holonomy* $\mathrm{hol}(\xi, \gamma) \in \mathbb{C}^\times$.

It is known that $H^2(X, \mathcal{D}^2)$ is the group of all isomorphism classes of line bundles on X with connection. In this case the connection determines a curvature and a holonomy which are the curvature and holonomy of the corresponding Deligne class.

2.2. Transgression

Let $\mathrm{ev} : S^1 \times L(X) \to X$ be the evaluation map and recall that there is a transgression map

$$\tau : \Omega^{p+1}(X) \to \Omega^p(L(X)),$$

defined as follows. If $F \in \Omega^{p+1}(X)$ then $\tau(F)$ is the result of pulling back F with ev to $S^1 \times L(X)$ and then integrating over the circle. There is an analogous map

$$\tau : H^{p+1}(X, \mathbb{Z}) \to H^p(L(X), \mathbb{Z}).$$

2.3. Deligne class of a torsion class

Next we need a result about torsion classes (Section 3.5). Let $\mathbb{Z}_d \subset U(1)$ be the group of dth roots of unity. Then to any class $\mu \in H^p(X, \mathbb{Z}_d)$ there is a Deligne class $\alpha(\mu) \in H^p(X, \mathcal{D}^p)$. The class of $\alpha(\mu)$ is the image of μ under the Bockstein map $H^p(X, \mathbb{Z}_d) \to H^{p+1}(X, \mathbb{Z})$ induced by the short exact sequence

$$\mathbb{Z} \xrightarrow{\times d} \mathrm{d}\mathbb{Z} \xrightarrow{\exp(2\pi i)} \mathbb{Z}_d.$$

2.4. Line bundle on loop space

For every Deligne class ξ in $H^2(X, \mathcal{D}^2)$ there is a line bundle $L_\xi \to L(X)$ over the loop space of X (Section 3.6). This correspondence is essentially a homomorphism, that is $L_{\xi+\eta} = L_\xi \otimes L_\eta$, $L_{-\xi} = L_\xi^*$ and $L_0 = U(1) \times L(X)$. It is important to note here that these really are equalities in the sense that there are canonical isomorphisms in each case. The Chern class of L_ξ is the transgression of the class of ξ.

There is also a natural connection on $L_\xi \to L(X)$ whose curvature is the transgression of F_ξ and whose holonomy is determined as follows. If $\gamma : S^1 \times L(X)$ then the holonomy around γ is the holonomy of ξ around ev $\circ$ id $\times \gamma$, where id $\times \gamma : S^1 \times S^1 \to S^1 \times L(X)$ is the map (id $\times \gamma)(\theta, \phi) = (\theta, \gamma(\phi))$.

2.5. Sections of the line bundle on loop space

We are interested in sections of the line $L_\xi \to L(X)$ and its pullback to $\Sigma(X)$.

The first of these arises because there is a canonical non-vanishing section (trivialisation)

$$\phi_\xi : \Sigma(X) \to \partial^{-1}(L_\xi)$$

defined below in Eq. (3.13).

The second case is when $c(\xi) = 0$. Then the transgression of $c(\xi)$ is zero and hence $c(L_\xi) = 0$. It follows that L_ξ is trivial or admits a global non-vanishing section. But now there is not a canonical section. However if we choose a ρ with $\iota(\rho) = \xi$ then we can construct a section

$$\chi_\rho : L(X) \to L_\xi.$$

Notice that, from the exact sequence of groups (3.8) mentioned above if we change ρ to τ with $\iota(\tau) = \xi$ then $\tau - \rho$ is a closed two-form whose integral over any two surface is an integral multiple of $2\pi i$. The two-form τ also defines a section χ_τ of L_ξ so we must have that $\chi_\rho = w\chi_\tau$ for some function $w : L(X) \to U(1)$. The function w is defined as follows. If σ is a map of a disk D into X with boundary a loop γ the function

$$w(\gamma) = \exp\left(\int_D \sigma^*(\tau - \rho)\right)$$

is well-defined and independent of the choice of σ. This construction is, of course, just the definition of the Wess–Zumino–Witten action of $\tau - \rho$. We will see in Sections 4.8 and 4.9 how to understand this fact in terms of gauge transformations of bundle gerbes.

With these observations we can construct world sheet actions. We start with the following result of Freed and Witten [6]. The theory of elliptic operators can be used to construct a line bundle $J_Q \to L(Q)$ with a section

$$\text{Pfaff} : \Sigma_Q(M) \to \partial^{-1}(J_Q).$$

Let $w_2 \in H^2(Q, \mathbb{Z}_2)$ be the second Stiefel–Whitney class of the normal bundle of Q. This is a torsion class so $\alpha(w_2)$ is a Deligne class in $H^3(Q, \mathcal{D}^3)$.

The line bundle J_Q has Chern class the transgression of $c(\alpha(w_2))$ and a natural flat connection whose holonomy along $\gamma : S^1 \to Q$ is given by (id $\times \gamma)^*(w_2)$. It follows from

the discussion above that $L_{\alpha(w_2)}$ with its natural connection is isomorphic to J_Q and has the same holonomy. Hence we can regard Pfaff as a

$$\text{Pfaff} : \Sigma_Q(M) \to \partial^{-1}(L_{\alpha(w_2)}). \tag{2.2}$$

up to a choice of a constant depending on the D-brane Q.

2.6. Case 1

Assume that the B-field or equivalently the Deligne two class ξ it defines (see Example 3.3) on all of X is such that

$$c(\xi_{|Q}) = c(\alpha(w_2)). \tag{2.3}$$

It follows that $L_{\xi_{|Q}} \to L(Q)$ and $L_{\alpha(w_2)} \to L(Q)$ are isomorphic over $L(Q)$. By choosing some $\rho \in \Omega^2(Q)$ with $\iota(\rho) = \xi_{|Q} - \alpha(w_2)$ we obtain a non-vanishing section χ_ρ of $L_\xi \otimes L^*_{\alpha(w_2)}$. Finally notice that $\partial^{-1}(L_{-\xi}) \to \Sigma(X)$ has a non-vanishing section $\phi_{-\xi}$ over $\Sigma(X)$ and this restricts to a non-vanishing section $\phi_{-\xi}$ over $\Sigma_Q(X)$. We can now put all the pieces together. The tensor product

$$W(\rho, \xi) = \text{Pfaff} \otimes \partial^{-1}(\chi_\rho) \otimes \phi_{-\xi} \tag{2.4}$$

(where $\partial^{-1}(\chi_\rho)$ denotes the pullback of the section χ_ρ) is a section of $\partial^{-1}(L_{\alpha(w_2)} \otimes L_{\xi_{|Q}} \otimes L^*_{\alpha(w_2)} \otimes L^*_{\xi_{|Q}})$ and hence is a function on $\Sigma_Q(M)$. In [6] χ_ρ is regarded as a kind of connection (it is their A-field). Notice that if we change ρ to τ subject to requiring that $\iota(\tau) = \xi_{|Q} - \alpha(w_2)$ and $\sigma \in \Sigma(M)$ then we have (using $*$ to denote pullback of forms):

$$W(\rho, \xi)(\sigma) = W(\tau, \xi)(\sigma)w(\tau - \rho)(\partial^*\sigma). \tag{2.5}$$

2.7. Case 2

The B-field is torsion on restriction to Q but (2.3) does not hold, that is, the difference between $c(\xi_{|Q})$, which comes from the B-field, and $c(\alpha(w_2))$ is non-zero.

In this case, in order to cancel the anomaly we need an auxiliary geometric structure. In [12] Azumaya algebras played this role. Here we use bundle gerbes, bundle gerbe modules and connections on these to give ingredients that we can feed into the world sheet action to cancel the anomaly which is essentially

$$c(\xi_{|Q}) - c(\alpha(w_2)). \tag{2.6}$$

We will show in Section 4.1 that any bundle gerbe with connection and curving gives rise to a Deligne two class. If this Deligne class is torsion the bundle gerbe admits so-called bundle gerbe modules. If A is a connection on a bundle gerbe module for a bundle gerbe with Deligne class η over a manifold X then we prove in Section 4.11 that the trace of the holonomy of A defines a section tr hol(A) of $L_\eta \to L(X)$. This section is an extra ingredient that may be used in forming world sheet actions.

Kapustin [12] on the other hand considers a $PU(n)$ bundle $P \to Q$ with class $\zeta \in H^2(Q, \mathbb{Z}_n)$ and an Azumaya algebra module connection A on $P \times \mathbb{C}^n$. As ζ defines a Deligne

class it defines a line bundle $L_\zeta \to L(Q)$. The trace of the holonomy of A is a section of this line bundle and hence pulls back to give a section $\partial^{-1}(\mathrm{tr}\,\mathrm{hol}(A))$ of $\partial^{-1}(L_\zeta) \to \Sigma(X)$. The product of the Pfaffian of the Dirac operator and the pullback of the trace of the holonomy of A is now a section of $\partial^{-1}(L_{\alpha(w_2)} \otimes L_\zeta)$ and to make this trivial Kapustin assumes that ζ can be chosen so that

$$c(\alpha(w_2)) + c(\zeta) = [H]_Q,$$

where $[H] \in H^3(X, \mathbb{Z})$ is the three-class arising from the Bockstein map applied to the B-field. It follows that we can trivialise the line bundle $L_{\alpha(w_2)} \otimes L_\zeta \otimes L_\xi^* \to L(Q)$ (recall that L_ξ is the line bundle arising from the Deligne class ξ or equivalently, from the B-field).

Choosing a ρ with $\iota(\rho) = \xi|_Q - \zeta - \alpha(w_2)$ we obtain a non-vanishing section χ_ρ of $L_\xi \otimes L_\zeta^* \otimes L_{\alpha(w_2)}^*$ We then obtain an action by generalising the construction (2.4) to this situation.

The bundle gerbe version of this is as follows. Start with the torsion class $\alpha(w_2) - \xi|_Q$ on Q. *Define* ζ to be $\alpha(w_2) - \xi|_Q$. There is an associated lifting bundle gerbe with Dixmier–Douady class $c(\zeta) = c(\alpha(w_2)) - c(\xi|_Q)$ (this is described in Section 4.3). A bundle gerbe module for this lifting bundle gerbe is just a $PU(n)$ bundle $P \to Q$ for some integer n (Section 4.9). This is the connection with Kapustin's approach and we can proceed by analogy with [12]. Choose a bundle gerbe module connection A on P. We will show (Section 4.11) that the trace of the holonomy of A is a section of $L_\zeta \to L(Q)$.

Choosing a stable isomorphism of $L_{\xi|Q}$ and $L_{\alpha(w_2)} \otimes L_\zeta$ defines a section χ of $L_{\alpha(w_2)}^* \otimes L_\zeta^* \otimes L_{\xi|Q}$. The total world sheet action is then

$$\mathrm{Pfaff} \otimes \partial^{-1}(\mathrm{tr}\,\mathrm{hol}(A)) \otimes \phi_{-\xi} \otimes \partial^{-1}(\chi). \tag{2.7}$$

Note that in [12] the χ dependence of the action is suppressed.

2.8. Case 3

The B-field is not torsion on restriction to Q.

We can proceed as in Case 2 up until we find that the bundle gerbe module for the lifting bundle gerbe over Q has to have fibre an infinite dimensional Hilbert space $\mathcal{H}$. Connections on such a module take their values in the compact operators on $\mathcal{H}$ and so cannot have trace class holonomy. Following Section 9 of [2] we observe that if there are bundle gerbe connections taking values in the trace class operators on $\mathcal{H}$ then the difference of the holonomy of two of these (say A_1 and A_2) is trace class. So we fix a reference bundle gerbe module connection A_1 taking values in the trace class operators on $\mathcal{H}$. If A_2 is any other trace class operator valued bundle gerbe connection we will show (Section 4.11) that $\mathrm{tr}(\mathrm{hol}(A_1) - \mathrm{hol}(A_2))$ is a well-defined section of $L_\zeta \to L(Q)$. Then the world sheet action is the function

$$\mathrm{Pfaff} \otimes \partial^{-1}[\mathrm{tr}(\mathrm{hol}(A_1) - \mathrm{hol}(A_2))] \otimes \phi_{-\xi} \otimes \partial^{-1}(\chi). \tag{2.8}$$

In the remainder of this paper we discuss the mathematics behind all these constructions. We begin with the standard description of Deligne cohomology in terms of double com-

plexes (hyper-cohomology). We then pass to the description of Deligne cohomology in terms of differential characters. This has a number of advantages over the double complex point of view. In particular it is a global description not requiring an open cover to be chosen and moreover it is a precise description in the sense that the differential character is exactly the Deligne class rather than a representative of it in some cohomology theory.

3. Deligne cohomology

3.1. Local description

In this subsection we review the definition of Deligne cohomology before considering the anomaly cancellation argument. We let X be a general manifold for the purposes of this discussion noting that in most cases we will specialise to $X = Q$. Recall that for any positive integer q we have the exact sequence of sheaves $\mathcal{D}^q$ defined by

$$\underline{U(1)} \overset{d\log}{\longrightarrow} \Omega^1 \to \cdots \to \Omega^q \tag{3.1}$$

where $\underline{U(1)}$ is the sheaf of smooth functions with values in $U(1)$ and Ω^p is the sheaf of p-forms. We will define Deligne cohomology in terms of the sequence $\mathcal{D}^q$ below and use the notation $H^p(X, \mathcal{D}^q)$ for these groups although we shall be interested in the special case $q = p$, that is $H^p(X, \mathcal{D}^p)$.

Let $\mathcal{U} = \{U_\alpha\}_{a \in I}$ be a good open cover of X, that is every finite intersection of elements of $\mathcal{U}$ is contractible. We realise the disjoint union of all the open sets as

$$Y_\mathcal{U} = \{(x, \alpha) | x \in U_\alpha\} \tag{3.2}$$

and let $\pi : Y_\mathcal{U} \to X$ be the map $\pi(x, \alpha) = x$. The p-fold fibre product of $Y_\mathcal{U}$ with itself, over the map π is

$$Y_\mathcal{U}^{[p]} = \{(x, (\alpha_1, \alpha_2, \dots, \alpha_p)) | x \in U_{\alpha_1} \cap \cdots \cap U_{\alpha_p}\} \subset X \times I^p \tag{3.3}$$

which is the disjoint union of all the p-fold intersections $U_{\alpha_1} \cap \cdots \cap U_{\alpha_p}$. We define projection maps $\pi_i : Y_\mathcal{U}^{[p]} \to Y_\mathcal{U}^{[p-1]}$ for each $i = 1, \dots, p$ by $\pi_i(x, (\alpha_1, \dots, \alpha_p)) = (x, (\alpha_1, \dots, a_{i-1}, a_{i+1}, \dots, a_p))$ and a map $\delta : \Omega^r(Y_\mathcal{U}^{[p]}) \to \Omega^r(Y_\mathcal{U}^{[p-1]})$ by

$$\delta = \sum_{i=1}^{p} (-1)^i \pi_i^*.$$

The space $\Omega^p(Y_\mathcal{U}^{[q]})$ is the usual space of p-form valued cocycles and the map δ is the usual coboundary map for Cech cohomology. If $\omega \in \Omega^p(Y_\mathcal{U}^{[q]})$ we let $\omega_{\alpha_1 \dots \alpha_p}$ denote the restriction of ω to $U_{\alpha_1} \cap \dots \cap U_{\alpha_p}$ in the usual way.

To calculate the Deligne cohomology we form the double complex:

$$
\begin{array}{cccccccc}
\vdots & & \vdots & & \vdots & & & \vdots \\
\delta \uparrow & & \delta \uparrow & & \delta \uparrow & & & \delta \uparrow \\
\underline{U(1)}(Y_{\mathcal{U}}^{[3]}) & \xrightarrow{d\log} & \Omega^1(\Omega^{[3]}) & \xrightarrow{d} & \Omega^2(Y_{\mathcal{U}}^{[3]}) & \xrightarrow{d} & \cdots \xrightarrow{d} & \Omega^q(Y_{\mathcal{U}}^{[3]}) \\
\delta \uparrow & & \delta \uparrow & & \delta \uparrow & & & \delta \uparrow \\
\underline{U(1)}(Y_{\mathcal{U}}^{[2]}) & \xrightarrow{d\log} & \Omega^1(Y_{\mathcal{U}}^{[2]}) & \xrightarrow{d} & \Omega^2(Y_{\mathcal{U}}^{[2]}) & \xrightarrow{d} & \cdots \xrightarrow{d} & \Omega^q(Y_{\mathcal{U}}^{[2]}) \\
\delta \uparrow & & \delta \uparrow & & \delta \uparrow & & & \delta \uparrow \\
\underline{U(1)}(Y_{\mathcal{U}}) & \xrightarrow{d\log} & \Omega^1(Y_{\mathcal{U}}) & \xrightarrow{d} & \Omega^2(Y_{\mathcal{U}}) & \xrightarrow{d} & \cdots \xrightarrow{d} & \Omega^q(Y_{\mathcal{U}})
\end{array}
\tag{3.4}
$$

The real Deligne cohomology is the cohomology of the double complex ((3.4)) which is calculated by forming the 'diagonal' complex

$$
\underline{U(1)}(Y_{\mathcal{U}}) \xrightarrow{D} \underline{U(1)}(Y_{\mathcal{U}}^{[2]}) \oplus \Omega^1(Y_{\mathcal{U}}) \xrightarrow{D} \underline{U(1)}(Y_{\mathcal{U}}^{[3]}) \oplus \Omega^2(Y_{\mathcal{U}}^{[2]}) \oplus \Omega^3(Y_{\mathcal{U}}) \xrightarrow{D} \cdots
\tag{3.5}
$$

where the maps D are defined recursively by (for $g \in \underline{U(1)}(Y_{\mathcal{U}})$)

$$
\begin{aligned}
D(g) &= (\delta(g), d\log g) = (\delta(g), g^{-1}\,dg), \\
D(g, \omega^1) &= (\delta(g), \delta(\omega^1) - g^{-1}\,dg, \, d\omega^1), \\
D(g, \omega^1, \omega^2) &= (\delta(g), \delta(\omega^1) + g^{-1}\,dg, \delta(\omega^2) - d\omega^1, \, d\omega^2). \\
\vdots &
\end{aligned}
$$

Standard results in sheaf theory can be applied to show that the cohomology of the complex (3.5) is independent of the choice of good cover. Similarly we can show that if $f : X \to N$ is a smooth map then we have a pull-back map

$$
f^* : H^p(N, \mathcal{D}^q) \to H^p(X, \mathcal{D}^q)
$$

on Deligne cohomology.

We are interested in the particular case when $p = q$. Then a Deligne class is determined by a collection

$$
(g, \omega^1, \ldots, \omega^q) \in \underline{U(1)}(Y_{\mathcal{U}}^{[q+1]}) \oplus \Omega^1(Y_{\mathcal{U}}^{[q]}) \oplus \cdots \oplus \Omega^q(Y_{\mathcal{U}})
$$

satisfying $D(g, \omega^1, \ldots, \omega^q) = 0$ or $\delta(g) = 1, \delta(\omega^1) = (-1)^{q-1} g^{-1}\,dg, \delta(\omega^2) = (-1)^{q-2}\,d\omega^1, \ldots, \delta(\omega^q) = d\omega^{q-1}$. Note that, from its definition as the cohomology of a complex, the Deligne class of $(g, \omega^1, \ldots, \omega^q)$ is unchanged if we replace it by

$$
\begin{aligned}
&(g, \omega^1, \ldots, \omega^q) + D(h, \mu^1, \ldots, \mu^{q-1}) \\
&= (g\delta(h), \omega^1 + (-1)^q h^{-1}dh + \delta(\mu^1), \omega^2 + (-1)^{q-1}d\mu^1 + \delta(\mu^2), \ldots, \omega^q + d\mu^{q-1})
\end{aligned}
\tag{3.6}
$$

where

$$
(h, \mu^1, \ldots, \mu^{q-1}) \in \underline{U(1)}(Y_{\mathcal{U}}^{[q]}) \oplus \Omega^1(Y_{\mathcal{U}}^{[q-1]}) \oplus \cdots \oplus \Omega^{q-1}(Y_{\mathcal{U}}).
\tag{3.7}
$$

Denote by $[g, \omega^1, \ldots, \omega^q]$ the Deligne class containing $(g, \omega^1, \ldots, \omega^q)$. Associated to a Deligne class $\xi = [g, \omega^1, \ldots, \omega^q]$ is a $p + 1$ form $d\omega^q$. It is clear from Eq. (3.7) that this depends only on ξ. Moreover $\delta(d\omega^q) = d\delta(\omega^q) = dd\omega^{q-1}$ so that this is a $p+1$ form defined globally on X. Denote this form by F_ξ and call it the $(q + 1)$ *curvature* of the Deligne class.

Example 3.1. If $p = 0$ then a Deligne class is a smooth map $f : X \rightarrow U(1)$ and the curvature is the one-form $f^*(d\theta)$.

Example 3.2. If $p = 1$ then a Deligne class ξ can be represented by an isomorphism class of line bundle with connection. The curvature of the Deligne class is the curvature of the connection.

Example 3.3. This is the instance we are mostly concerned with in this paper. If $p = 2$ then a Deligne class can be represented by a stable isomorphism class of a bundle gerbe with connection and curving as reviewed in Section 4.6 and originally proved in [17]. As explained in [16] and reviewed in Section 4.5 a bundle gerbe with connection and curving gives rise to a three-curvature on the manifold X. The curvature of the Deligne class is precisely this three-curvature.

The B-field in string theory may be identified with the third component (ω^2) of a representative (g, ω^1, ω^2) of a Deligne class in $H^2(X, \mathcal{D}^2)$. The curvature of the Deligne class is called the H-field in string theory.

3.2. Holonomy of a Deligne class

Associated to any Deligne class

$$\xi = [g, \omega^1, \ldots, \omega^p]$$

is a cohomology class $c(\xi) = [g]$ in $H^{p+1}(X, \mathbb{Z})$. The image of $c(\xi)$ in real cohomology is the class of $(1/2\pi i) F_\xi$. Let us call the Deligne class ξ trivial if the Chern class $c(\xi)$ is zero. Note this is not the same as the Deligne class being zero. If $\rho \in \Omega^p(X)$ then we can restrict it to each open set or equivalently pull it back to $Y_\mathcal{U}$ and hence determine a form $\pi^*(\rho)$. This determines a Deligne p class $\iota(\rho) = [1, 0, \ldots, 0, \pi^*(\rho)]$ which is clearly trivial. Hence we have a sequence of maps

$$\Omega^p(X) \xrightarrow{\iota} H^p(X, \mathcal{D}^p) \xrightarrow{c} H^{p+1}(X, \mathbb{Z})$$

with $c \circ \iota = 0$. Let $\Omega^p(X)_{(c,0)}$ denote the subset of p-forms which are closed and whose class in $H^p(X, \mathbb{R})$ is the image of a class from $H^p(X, 2\pi i\mathbb{Z})$. Then there is an exact sequence

$$0 \rightarrow \Omega^p(X)_{(c,0)} \rightarrow \Omega^p(X) \xrightarrow{\iota} H^p(X, \mathcal{D}^p) \xrightarrow{c} H^{p+1}(X, \mathbb{Z}) \rightarrow 0. \tag{3.8}$$

Assume that X is p-dimensional so that $H^{p+1}(X, \mathbb{Z}) = 0$. Then every Deligne class ξ is trivial so $\xi = \iota(\rho)$ for some form ρ on X and, assuming that X is oriented, we can define

$$\text{hol}(\xi, X) = \exp \int_X \rho.$$

If we choose another ρ' with $\iota(\rho') = \xi$ then

$$\int_X (\rho - \rho') \in 2\pi i \mathbb{Z},$$

so that $\mathrm{hol}(\xi, X)$ is independent of the choice of ρ. Notice that $F_\xi = \mathrm{d}\rho$ so if X is the boundary of a $p + 1$-dimensional manifold Y then we have

$$\mathrm{hol}(\xi, X) = \exp \int_{\partial Y} \rho, \tag{3.9}$$

$$= \exp \int_Y F_\xi \tag{3.10}$$

where $X = \partial Y$ has the induced orientation.

More generally if X is not necessarily p-dimensional, we can consider a map $\gamma : \Sigma^p \to X$, where Σ^p is p-dimensional and compact and define

$$\mathrm{hol}(\xi, \gamma) = \mathrm{hol}(\gamma^*(\xi), \Sigma^p).$$

Similarly if X^{p+1} is a p-dimensional oriented manifold with boundary $\partial X^{p+1} = \Sigma^p$, a p-dimensional manifold, and $\gamma : X^{p+1} \to X$ we have

$$\mathrm{hol}(\xi, \partial \gamma) = \exp \left(\int_{X^{p+1}} \gamma^*(F_\xi) \right),$$

where $\partial \gamma : \Sigma^p \to X$ is the restriction of γ to the boundary.

Example 3.4. If $p = 0$ then a Deligne class is a smooth map $f : X \to U(1)$ and the one-form associated to the class is $f^*(\mathrm{d}\theta)$. The holonomy of the smooth map is over a point p and is just the evaluation of f at p.

Example 3.5. If $p = 1$ then a Deligne class can be represented by an isomorphism class of line bundle with connection. The holonomy is the classical holonomy of a connection.

Example 3.6. If $p = 2$ then a Deligne class can be represented by a stable isomorphism class of a bundle gerbe with connection and curving. The holonomy is the holonomy of a connection and curving defined in [16] and reviewed in Section 4.6.

Using (3.9) we can define the gluing property of holonomy. Let Σ_i for $i = 1, 2, 3$ be manifolds of dimension p related as follows. Assume we have open sets $U_i \subset \Sigma_i$ for $i = 1, 2$ such that $\Sigma_i - U_i$ and U_i are manifolds with (common) boundary. Moreover assume we have an orientation reversing diffeomorphism $\phi : U_1 \to U_2$ of manifolds with boundary so that $\partial \phi : \partial(\Sigma_1 - U_1) \to \partial(\Sigma_2 - U_2)$ is a diffeomorphism. Finally assume that Σ_3 is the manifold constructed by using $\partial \phi$ to glue together $\Sigma_1 - U_1$ and $\Sigma_2 - U_2$. Consider now a pair of maps $\gamma_i : \Sigma_i \to X$ such that $\gamma_2|_{U_2} \circ \phi = \gamma_1|_{U_1}$. Then there is an induced map $f_1 \# f_2 :: \Sigma_3 \to X$. This map may not be smooth on the common boundary of the $\Sigma_i - U_i$ but we can still define its holonomy. Then we have the following proposition.

Proposition 3.7 (Holonomy gluing property). *In the situation above*

$$\mathrm{hol}(\xi, \gamma_1 \# \gamma_2) = \mathrm{hol}(\xi, \gamma_1) \, \mathrm{hol}(\xi, \gamma_2).$$

Proof. Using the definition of holonomy as an integral it is easy to see that

$$
\begin{aligned}
\mathrm{hol}(\xi, \gamma_1 \# \gamma_2) &= \mathrm{hol}(\xi, \gamma_1|_{(\Sigma_1 - U_1)}) \, \mathrm{hol}(\xi, \gamma_2|_{(\Sigma_1 - U_2)}) \\
&= \mathrm{hol}(\xi, \gamma_1|_{(\Sigma_1 - U_1)}) \, \mathrm{hol}(\xi, \gamma_1|_{U_1}) \, \mathrm{hol}(\xi, \gamma_1|_{U_1})^{-1} \, \mathrm{hol}(\xi, \gamma_2|_{(\Sigma_1 - U_2)}) \\
&= \mathrm{hol}(\xi, \gamma_1|_{(\Sigma_1 - U_1)}) \, \mathrm{hol}(\xi, \gamma_1|_{U_1}) \, \mathrm{hol}(\xi, \gamma_2|_{U_2}) \, \mathrm{hol}(\xi, \gamma_2|_{(\Sigma_1 - U_2)}) \\
&= \mathrm{hol}(\xi, \gamma_1) \, \mathrm{hol}(\xi, \gamma_2).
\end{aligned}
$$

Here we use the fact that ϕ is orientation reversing to deduce that

$$\mathrm{hol}(\xi, \gamma_1|_{U_1})^{-1} = \mathrm{hol}(\xi, \gamma_2|_{U_2}). \qquad \square$$

A remark may help the reader to visualise the gluing here when $p = 2$. Imagine that Σ_1 and Σ_2 are two balloons that are pressed together so they touch on an open disk $U_1 = U_2$. Cut out the region where the balloons meet and we obtain the surface Σ_3. We are suppressing mention here of the inclusion maps f_1 and f_2 of the surfaces into $X = \mathbb{R}^3$. Notice that it would be easier to state the (3.1) as the holonomy of U_1 times the holonomy over $\Sigma_1 - U_1$ equals the holonomy over Σ_1 but we cannot as holonomy is only defined for closed surfaces.

3.3. Local formulae

To compare with the calculations in [6] it is useful to have a local formulation of the holonomy. We will restrict attention to a Deligne two class although a general formula is possible. Formulae of this type have appeared previously in the work of Gawedzki [8,9], Brylinski [3], and Kapustin [12] for Deligne classes of arbitrary degree in [10]. In these applications the formulae were used to define the holonomy, here we have an intrinsic definition and we will derive the local formula. The case of a Deligne class of arbitrary degree is in [19].

Consider then a Deligne two class $\xi = [g, k, B]$ relative to an open cover $\{U_\alpha\}$ of X. We pull this class back to a surface Σ without boundary via a map $\sigma : \Sigma \to X$ and obtain the class $\sigma^*(\xi) = [\sigma^*(g), \sigma^*(k), \sigma^*(B)]$ relative to the open cover $\{\sigma^{-1}(U_\alpha)\}$ of Σ. As Σ is two-dimensional this class is trivial and we have

$$\sigma^*(g_{\alpha\beta\gamma}) = h_{\beta\gamma} h_{\alpha\gamma}^{-1} h_{\alpha\beta},$$

and we can find m_α such that

$$\sigma^*(k_{\alpha\beta}) + h_{\alpha\beta}^{-1} \, \mathrm{d}h_{\alpha\beta} = m_\beta - m_\alpha.$$

If follows that $\sigma^*(B)_\alpha^{-1} - \mathrm{d}m_\alpha$ is a globally defined two-form the exponential of whose integral over Σ is the holonomy.

Assume now that we have a triangulation of Σ into faces, edges and vertices which is subordinate to the open cover $\{\sigma^{-1}(U_\alpha)\}$. That is the closure of each face is in (at least

one) open set. For each face f we choose a particular open set $\sigma^{-1}(U_\rho(f))$ such that $f \subset \sigma^{-1}(U_\rho(f))$. Similarly for each edge e and vertex v. Then we have

$$\text{hol}(\Sigma, \xi) = \prod_f \exp\left(\int_f \sigma^*(B_{\rho(f)}) - \mathrm{d}m_{\rho(f)}\right),$$

where we orient each face with the orientation it inherits from Σ. Using Stoke's theorem this becomes

$$\text{hol}(\Sigma, \xi) = \prod_f \exp\left(\int_f \sigma^*(B_{\rho(f)})\right) \prod_{e \subset f} \exp\left(-\int_e m_{\rho(f)}\right),$$

where the second product is over all pairs (e, f) consisting of an edge contained in a face. In the integral the edge is oriented by the face. For a pair $e \subset f$ we have

$$-m_{\rho(f)} = -m_{\rho(e)} + \sigma^*(k_{\rho(f)\rho(e)}) + h^{-1}_{\rho(f)\rho(e)}\mathrm{d}h_{\rho(f)\rho(e)}.$$

Notice that

$$\sum_{e \subset f} \int_e -m_{\rho(e)},$$

vanishes as every edge occurs in exactly two faces and with opposite orientations. We use here the fact that Σ is a manifold without boundary. Hence we have, again using Stoke's theorem, that

$$\text{hol}(\Sigma, \xi) = \prod_f \exp\left(\int_f \sigma^*(B_{\rho(f)})\right) \prod_{e \subset f} \exp\left(\int_e \sigma^*(k_{\rho(f)\rho(e)})\right) \prod_{v \subset e \subset f} h_{\rho(f)\rho(e)}(v).$$

For a triple $v \subset e \subset f$ we have

$$h_{\rho(f)\rho(e)}(v) = \sigma^*(g_{\rho(f)\rho(e)\rho(v)})(v)h_{\rho(f)\rho(v)}(v)h^{-1}_{\rho(e)\rho(v)}(v),$$

and substituting again and observing that the remaining h terms cancel we obtain

$$\text{hol}(\Sigma, \xi) = \prod_f \exp\left(\int_f \sigma^*(B_{\rho(f)})\right)$$
$$\times \prod_{e \subset f} \exp\left(\int_e \sigma^*(k_{\rho(f)\rho(e)})\right) \prod_{v \subset e \subset f} \sigma^*(g_{\rho(f)\rho(e)\rho(v)}(v)). \tag{3.11}$$

3.4. Differential characters

We have seen that we can construct from a Deligne cohomology class ξ of degree p a holonomy operation and a curvature form F_ξ which satisfy holonomy gluing (Proposition 3.7) and the relation in Eq. (3.9). In an appropriate sense these two data determine the Deligne cohomology class exactly. The appropriate sense is the theory of *differential characters*. A differential character [3,4] is a pair (h, F), where h is a homomorphism from $Z_p(X)$, the

group of all smooth, closed p chains (cycles) in X, to $U(1)$ and F is a $p + 1$ form. These two are required to be related by

$$h(\partial\mu) = \exp\left(\int_\mu F\right)$$

for any $p + 1$ chain μ (cf. (3.9)). The homomorphism condition on h can be interpreted as the holonomy gluing condition. The set of all such differential characters is denoted by $\hat{H}^p(X, U(1))$.

Our construction of holonomy and curvature of a Deligne class has essentially[1] defined a map $H^p(X, \mathcal{D}^p) \to \hat{H}^p(X, U(1))$ and it is a result of [4] that these two spaces are, in fact, isomorphic. In the remainder of this paper we shall work primarily with differential characters as our representation for Deligne cohomology. Because of this isomorphism we can reinterpret various maps we have defined for Deligne cohomology in terms of differential characters. First notice that the curvature of a differential character $\xi = (h, F)$ is, of course, F.

Secondly the map

$$\iota : \Omega^p(X) \to \hat{H}^p(X, U(1))$$

is defined as follows. Let $h_\rho : Z_p(X) \to U(1)$ be defined by $h_\rho(\sigma) = \exp(\int_\sigma \rho)$ for any $\rho \in \Omega^p(X)$. This is a homomorphism and we let $\iota(\rho) = (h_\rho, \mathrm{d}\rho)$.

Thirdly there is an induced map $c : \hat{H}^p(X, U(1)) \to H^p(X, \mathbb{Z})$. We follow the discussion in [3]. Let $C_p(X)$ be the group of all chains. Results from group theory imply that there is a map $\hat{h} : C_p(X) \to \mathbb{R}$ such that $h(\sigma) = \exp(\hat{h}(\sigma))$ for any $\sigma \in C_p(X)$. Then

$$\int_\mu F - \hat{h}(\partial\mu) \in 2\pi i\mathbb{Z}$$

for any $\mu \in C_{p+1}(X)$. Let $\tau(\mu) = (1/2\pi i)(\int_\mu F - \hat{h}(\partial\mu))$ and notice that $\partial^*(\tau) = 0$ so that $[\tau] \in H^{p+1}(X, \mathbb{Z})$. It is straightforward to check that changing the choice of $\hat{h}$ does not change the class of $[\tau]$ and we define $c(h, F) = [\tau]$.

Our preference for differential characters is due to their mathematical simplicity and a belief that they are generally the observable quantities in D-brane physics. However there are many situations where we want to work with geometric objects which determine a Deligne class rather than with representatives of the Deligne classes or of the differential characters themselves. For any p there are a number of such geometric objects, for example for $p = 2$, the case of interest in this note, there are gerbes, bundle gerbes, local gerbes in the sense of Hitchin, $\mathbb{Z}$ bundle two gerbes and BS^1 bundles in the sense of Gajer [7]. All of these, when endowed with appropriate notions of connection and curvature, determine degree two Deligne classes and differential characters. While we have a bias towards bundle gerbes (evident later in this article) the formalism for D-branes incorporates additional structure beyond what we

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[1] This is not completely true as we have defined holonomy only over cycles which arise as the images of maps of triangulated manifolds. We will ignore this issue for the remainder of the discussion as it does not affect what we are doing.

have described here. It is conceivable that one of these geometric realisations will be preferable when this additional structure is taken into account, however our point here is that since we can formulate the discussion in terms of Deligne characters, and all the geometric realisations lead to these, our account is independent of whatever geometric realisation is chosen.

It is not clear which physical applications can motivate a preference for one of these geometric realisations over another. We will not attempt to comment further on this question here.

3.5. Deligne class of a torsion cohomology class

Recall that the short exact sequence of groups

$$\mathbb{Z} \overset{\times d}{\to} \mathbb{Z} \to \mathbb{Z}_d,$$

induces the Bockstein map $\beta : H^p(X, \mathbb{Z}_d) \to H^{p+1}(X, \mathbb{Z})$. We will show that there is a map $\eta : H^p(X, \mathbb{Z}_d) \to H^p(X, \mathcal{D}^p)$ such that $c \circ \eta = \beta$.

Let $\kappa \in H^p(X, \mathbb{Z}_d)$. Choose a representative $r \in \kappa$. Then r is a homomorphism from $C^p(X)$, the group of all p chains, into $\mathbb{Z}_d$. We can restrict this to obtain a homomorphism from $Z_p(X)$ into $\mathbb{Z}_d$. If we choose another representative r' of κ then $(r - r') = \partial^*(s)$ for some $s \in C_{p-1}(X)$ so that if σ is a closed p chain then $\langle r, \sigma \rangle - \langle r', \sigma \rangle = \langle r - r', \sigma \rangle = \langle \partial^*(s), \sigma \rangle = \langle s, \partial(\sigma) \rangle = 0$. So we have a well-defined homomorphism $h_\kappa : Z_p(X) \to \mathbb{Z}_d \subset U(1)$. If $\sigma = \partial(\tau)$ then $h_\kappa(\sigma) = \langle r, \partial(\tau) \rangle = \langle \partial^*(r), \tau \rangle = 0$. Hence the pair $(h_\kappa, 0)$, where 0 is the zero $(p+1)$-form defines a Deligne cohomology class we denote by $\eta(\kappa)$. In terms of Cech representatives relative to an open cover we can represent κ as the $\mathbb{Z}_d \subset U(1)$ valued cocycle $\kappa_{i_0,\dots,i_p}$ for which $d\kappa_{i_0,\dots,i_p} = 0$ so that $\eta(\kappa) = (\kappa_{i_0,\dots,i_p}, 0, \dots, 0)$ defines a Deligne class. It is straightforward to check that $c(\eta(\kappa)) = \beta(\kappa)$ the image of κ under the Bockstein map.

3.6. Line bundles on loop space

Let X be a manifold of dimension m and S a compact manifold of dimension p. Consider the evaluation map

$$\mathrm{ev} : S \times \mathrm{Map}(S, X) \to X.$$

If ρ is a differential $r + 1$ form on X then we can integrate its pull-back under ev to obtain an $r - p + 1$ form $\mathrm{ev}_*(\rho)$ on $\mathrm{Map}(S, X)$ called the *transgression* of ρ. This transgression operation can be extended to act on differential characters, and hence Deligne cohomology as follows. Let (h, F) be a differential character with $h : Z_r(X) \to U(1)$ and F an $r + 1$ form. Clearly we can transgress F to an $r + p - 1$ form on $\mathrm{Map}(S, X)$. Let $\sigma \in Z_{r-p}(S)$ and choose a class μ representing the generator of $H_p(S, \mathbb{Z}) = \mathbb{Z}$. Then $\mathrm{ev}_*(\sigma \times \mu) \in Z_r(X)$ and we can apply h. The result is a map $Z_{r-p}(\mathrm{Map}(S, X)) \to U(1)$ the transgression of F. It is, in fact, independent of the choice of representative μ and together with the transgression of F satisfies the conditions for a differential character. It is also possible to transgress a Cech representative for a Deligne class but the result is quite complicated and we refer the reader to [10] for details.

We concentrate now on the case that $S = S^1$ and hence Map(S, X) is the loop space $L(X)$ of all smooth maps of the circle into X. In this case the transgression of a Deligne two class ξ is a Deligne one class on $L(X)$ and hence it defines an isomorphism class of a line bundle and connection. We now give a geometric construction of this line bundle and connection. For convenience we assume that X is simply connected and let D be a disk with $D(X)$ the space of maps of D into X. Let $D(X)^{[2]}$ be pairs of maps $\sigma_1 : D \to X$ and $\sigma_2 : D \to X$ which agree on the boundary circle, that is $\sigma_{1|\partial D} = \sigma_{2|\partial D}$. Each such pair defines a map from the two sphere (thought of as the union of two copies of the disk) into X. Denote this map by $\sigma_1 \# \sigma_2$ and orient it by the first factor.

Let (h, F) be the differential character of a Deligne two class ξ. We define a line bundle $L_\xi \to L(X)$ whose fibre over a circle $\gamma : S^1 \to X$ is equivalence classes of pairs (σ, z) with $\partial(\sigma) = \gamma$ and $z \in \mathbb{C}$ and equivalence relation $(\sigma, z) \simeq (\sigma', z')$ if

$$h(\sigma \# \sigma')z = z'.$$

This means that a section of L_ξ is a function $s : D(X) \to \mathbb{C}$ such that

$$s(\sigma) = h(\sigma \# \sigma')s(\sigma'). \tag{3.12}$$

We think of this as a transformation rule just as tensor, spinor and gauge fields satisfy transformation rules for the Spin, Lorentz and gauge groups. In the case of sections of L_ξ there is no group but the philosophy is the same. Notice that this point of view has the advantage that sections are actually just functions, albeit on a larger space. In particular if $h = 1$ then the section transforms as $s(\sigma) = s(\sigma')$ and hence defines a function on $L(X)$.

The line bundle $L_\xi \to L(X)$ has a natural connection which we have no need in this paper to describe. Note however that if $\gamma : S^1 \to L(X)$ is a loop then it defines naturally a map $\tilde{\gamma} : S^1 \times S^1 \to X$ and the holonomy of the connection on L_ξ around γ is $h(\tilde{\gamma})$. It can be shown that the Deligne class of this line bundle with connection is the transgression of the Deligne two class on X.

3.7. Sections of the line bundle on loop space

In the construction of the world sheet action we need two basic sections of $L_\xi \to L(X)$ and its pull-back to $\Sigma(X)$ for a Deligne two class ξ on X. We first define these and then recall how they are used in Case 1.

Consider first

$$\phi_\xi : \Sigma(X) \to \partial^{-1}(L_\xi).$$

A section of the line bundle $\partial^{-1}(L_\xi) \to \Sigma(X)$ at a point $\nu \in \Sigma(X)$ is a function $s : D(X) \times_f \Sigma(X) \to \mathbb{C}$ satisfying $s(\sigma, \nu) = h(\sigma \# \sigma')s(\sigma', \nu)$, where elements of $D(X) \times_f \Sigma(X)$ are pairs $(\sigma, \nu) \in D(X) \times \Sigma(X)$ such that $\partial(\sigma) = \partial(\nu)$ and hence $\partial(\sigma) = \partial(\sigma')$. In particular, that the pullback $\partial^{-1}(L_\xi)$ has a canonical section defined by

$$\phi_\xi(\sigma, \nu) = h(\sigma \# \nu). \tag{3.13}$$

It follows from the holonomy gluing property that

$$\phi_\xi(\sigma, \nu) = h(\sigma \# \nu) = h(\sigma \# \sigma')h(\sigma' \# \nu) = h(\sigma \# \sigma')\phi_\xi(\sigma', \nu),$$

so that $\phi_\xi(\sigma, \nu)$ is indeed a section of $\partial^{-1}(L_\xi) \to \Sigma(X)$.

When we form an action out of tensors, spinors and gauge fields we must combine them so the resulting action transforms as a scalar. So too with world sheet actions. We must combine various sections of various bundles so that the final action transforms as $s(\sigma, \nu) = s(\sigma', \nu)$ and hence defines a function of σ.

Notice that if ξ and ξ' are two Deligne classes then $h_\xi h_{\xi'} = h_{\xi+\xi'}$. So if we multiply a section of L_ξ and a section of L'_ξ then it automatically transforms as a section of $L_{\xi+\xi'}$. This means we have canonical isomorphisms

$$L_\xi \otimes L_{\xi'} \to L_{\xi+\xi'}.$$

The other section used in the construction of the world sheet action is

$$\chi_\rho : L(X) \to L_\xi,$$

defined for a ρ with $\iota(\rho) = \xi$. To see how to define this we note that when $\iota(\rho) = \xi$ the holonomy and curvature of ξ are given by

$$h_{\iota(\rho)}(\sigma) = \exp\left(\int_\sigma \rho\right), \tag{3.14}$$

$$F_{\iota(\rho)} = \mathrm{d}\rho. \tag{3.15}$$

Note that $\exp(\int_\sigma \rho)$ and $\mathrm{d}\rho$ are unchanged if we add an integral, closed form to ρ, so as we expect depend only on $\iota(\rho) = \xi$ not on ρ. The section χ_ρ of $L_{\iota(\rho)}$ is defined by

$$\chi_\rho(\sigma) = \exp\left(\int_D \sigma^*(\rho)\right),$$

and it is easy to check that this satisfies $\chi_\rho(\sigma) = h_{\iota(\rho)}(\sigma \# \sigma') \chi_\rho(\sigma')$ as required for a section of $L_{\iota(\rho)}$. If we change ρ to $\rho + \mu$, where μ is a closed two-form whose integral over any closed surface is $2\pi i$ times an integer then

$$\chi_{\rho+\mu}(\sigma) = \exp\left(\int_D \sigma^*(\mu)\right) \chi_\rho(\sigma). \tag{3.16}$$

Recall how we apply these constructions to Case 1. We have the diagram (2.1)

$$\begin{array}{ccc} \Sigma(M) & \xrightarrow{\partial} & L(M) \\ \cup & & \cup \\ \Sigma_Q(M) & \xrightarrow{\partial} & L(Q) \end{array} \tag{3.17}$$

and we want to define a function on $\Sigma_Q(M)$. The ingredients are a Deligne two class (B-field) ξ on M and the (torsion) Steifel–Whitney class $w_2 \in H^2(Q, \mathbb{Z}_2)$ which together satisfy (2.3)

$$c(\xi_{|Q}) = c(\alpha(w_2)),$$

and the section

$$\text{Pfaff} : \Sigma_Q(M) \to \partial^{-1}(L_{\alpha(w_2)})$$

defined in (2.2).

First we apply the above constructions to get $\phi_{-\xi} : \Sigma(M) \to \partial^{-1}(L_{-\xi})$ and restrict this to $\Sigma_Q(M)$ to get (by abuse of notation)

$$\phi_{-\xi} : \Sigma_Q(M) \to \partial^{-1}(L_{-\xi})|_{\Sigma_Q(M)}.$$

Secondly, because $c(\xi_{|Q} - \alpha(w_2)) = 0$ we can choose $\rho \in \Omega^2(Q)$ such that $\iota(\rho) = c(\xi_{|Q}) = c(\alpha(w_2))$. Hence, applying the discussion above but replacing M by Q, we obtain a section $\chi_\rho : L(Q) \to L_\xi \otimes L^*_{\alpha(w_2)}$ and hence can pull this back to obtain a section

$$\partial^{-1}(\chi_\rho) : \Sigma_Q(M) \to \partial^{-1}(L_\xi) \otimes \partial^{-1}(L_{\alpha(w_2)})^*.$$

Combining these three sections we see that (2.5)

$$W(\rho, \xi) = \text{Pfaff} \otimes \partial^{-1}(\chi_\rho) \otimes \phi_{-\xi},$$

transforms in such a way that it is a function on $\Sigma_Q(M)$ which is the world sheet action.

4. A geometric interpretation

In this section we are interested in Cases 2 and 3 of Section 2 that is, general B-fields. We will use bundle gerbes to give a geometric interpretation of the Deligne character, transgression and the anomaly cancellation argument.

4.1. Bundle gerbes

Before defining bundle gerbes over a manifold X recall that if $\pi : Y \to X$ is a submersion (i.e. onto with onto differential) then X can be covered by open sets U_α such that there are sections $s_\alpha : U_\alpha \to X$ of π, that is $\pi \circ s_\alpha = 1$. A fibration is a submersion but not all submersions maps are fibrations. For example we can use the disjoint union $Y_{\mathcal{U}}$ of a given cover $\mathcal{U}$ as defined in (3.2). The sections are the maps $s_\alpha : U_\alpha \to Y_{\mathcal{U}}$ defined by $s_\alpha(x) = (x, \alpha)$.

Recall that a bundle gerbe[2] over X is a pair (L, Y), where $\pi : Y \to X$ is a submersion and L is a hermitian line bundle $P \to Y^{[2]}$ with a product, that is, a hermitian isomorphism

$$L_{(y_1, y_2)} \otimes L_{(y_2, y_3)} \to L_{(y_1, y_3)}$$

for every (y_1, y_2) and (y_2, y_3) in $Y^{[2]}$. We require the product to be smooth in y_1, y_2 and y_3 but in the interests of brevity we will not state the various definitions needed to make this requirement precise, they can be found in [16]. The product is required to be associative whenever triple products are defined. Also in [16] it is shown that the existence of the product and the associativity imply isomorphisms $L_{(y,y)} \simeq \mathbb{C}$ and $L_{(y_1, y_2)} \simeq L^*_{(y_2, y_1)}$.

If (L, Y) is a bundle gerbe we can define a new bundle gerbe, (L^*, Y), the dual of (L, Y), by taking the dual of L. Also if (L, Y) and (J, Z) are two bundle gerbes we can define their

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[2] Strictly speaking what we are about to define should be called a hermitian bundle gerbe but the extra terminology is overly burdensome.

product $(L \otimes J, Y \times_\pi Z)$, where $Y \times_\pi Z = \{(y, z) : \pi_Y(y) = \pi_Z(z)\}$ is the fibre product of Y and Z over their projection maps.

A morphism from a bundle gerbe (L, Y) to a bundle gerbe (J, Z) consists of a pair of maps (g, f), where $f : Y \to Z$ is a map commuting with the projection to X and $g : L \to J$ is a bundle map covering the induced map $f^{[2]} : Y^{[2]} \to Z^{[2]}$ and commuting with the bundle gerbe products on J and L respectively. If f and g are isomorphisms then we call (g, f) a bundle gerbe isomorphism.

If J is a (hermitian) line bundle over Y then we can define a bundle gerbe $\delta(J)$ by $\delta(J) = \pi_1^{-1}(J) \otimes \pi_2^{-1}(J)^*$, that is $\delta(J)_{(y_1, y_2)} = J_{y_2} \otimes J_{y_1}^*$, where $\pi_i : Y^{[2]} \to Y$ is the map which omits the ith element. The bundle gerbe product is induced by the natural pairing

$$J_{y_2} \otimes J_{y_1}^* \otimes J_{y_3} \otimes J_{y_2}^* \to J_{y_3} \otimes J_{y_1}^*.$$

A bundle gerbe which is isomorphic to a bundle gerbe of the form $\delta(J)$ is called *trivial*. A choice of J and a bundle gerbe isomorphism $\delta(J) \simeq L$ is called a *trivialisation*. If J and K are trivialisations of P then we have natural isomorphisms

$$J_{y_1} \otimes J_{y_2}^* \simeq K_{y_1} \otimes K_{y_2}^*,$$

and hence

$$J_{y_1}^* \otimes K_{y_1} \simeq J_{y_2}^* \otimes K_{y_2},$$

so that the bundle $J \otimes K$ is the pull-back of a hermitian line bundle on X. Moreover if J is a trivialisation and L is a bundle on X then $J \otimes \pi^{-1}(L)$ is also a trivialisation. Hence the set of all trivialisations of a given bundle gerbe is naturally acted on by the set of all hermitian line bundles on X.

One can think of bundle gerbes as one stage in a hierarchy of objects with each type of object having a characteristic class in $H^p(X, \mathbb{Z})$. For example if $p = 1$ we have maps from X to $U(1)$, the characteristic class is the pull-back of $\mathrm{d}z$. When $p = 2$ we have hermitian line bundles on X with characteristic class the Chern class. When $p = 3$ we have bundle gerbes and they have a characteristic class $\mathrm{d}(L) = \mathrm{d}(L, Y) \in H^3(X, \mathbb{Z})$, the Dixmier–Douady class of (L, Y). The Dixmier–Douady class is the obstruction to the bundle gerbe being trivial. In [16] the following theorem is proved.

Theorem 4.1. *A bundle gerbe (L, Y) has zero Dixmier–Douady class precisely when it is trivial.*

From [16] we also have the following proposition.

Proposition 4.2. *If L and J are bundle gerbes over X then*

(1) $\mathrm{d}(L^*) = -\mathrm{d}(L)$,
(2) $\mathrm{d}(L \otimes J) = \mathrm{d}(L) + \mathrm{d}(J)$.

We note finally that bundle gerbes behave nicely under pull-back. If (L, Y) is a bundle gerbe over X and $f : N \to X$ then we can pull-back Y and hence L to form a bundle gerbe $(f^{-1}(L), f^{-1}(Y))$ over N. We have $\mathrm{d}(f^{-1}(L), f^{-1}(Y)) = f^*(\mathrm{d}(L, Y))$.

4.2. Torsion bundle gerbes

The definitions of bundle gerbe, triviality and the Dixmier–Douady class can be immediately generalised with $U(1)$ replaced by any abelian group A except that the Dixmier–Douady class lives in $H^2(X, A)$. In particular we can consider bundle gerbes for any cyclic subgroup $\mathbb{Z}_d \subset U(1)$. The Dixmier–Douady class then lives in $H^2(X, \mathbb{Z}_d)$ and we call these torsion bundle gerbes or $\mathbb{Z}_d$ bundle gerbes.

It is natural to think of a torsion bundle gerbe as a $\mathbb{Z}_d$ subbundle of the $U(1)$ bundle $L \to Y^{[2]}$ which is stable under multiplication. The $U(1)$ bundle gerbe has Dixmier–Douady class in $H^3(X, \mathbb{Z})$ which is the Bockstein of the torsion bundle gerbe class in $H^2(X, \mathbb{Z}_d)$. Notice that there are two different notions of triviality for torsion bundle gerbes, the first is the vanishing of the class in $H^2(X, \mathbb{Z}_d)$ or torsion bundle gerbe triviality and the second is the vanishing of the associated $U(1)$ bundle gerbe or the vanishing of the class in $H^3(X, \mathbb{Z})$. The former implies the latter but not vice versa.

Standard results in topology tell us that every class in $H^3(X, Z)$ which is torsion arises as the Bockstein of a class in some $\mathbb{Z}_d$. Hence every bundle gerbe with torsion Dixmier–Douady class is stably isomorphic to a torsion bundle gerbe.

4.3. Lifting bundle gerbes

A common example of bundle gerbes is the so-called *lifting bundle gerbe*. Let

$$U(1) \to \hat{G} \xrightarrow{\pi} G \tag{4.1}$$

be a central extension of Lie groups and let $P \to X$ be a principal G bundle. Then there is a map $g : P^{[2]} \to G$ defined by $p_1 g(p_1, p_2) = p_2$. We can consider the central extension as a $U(1)$ bundle over G and pull it back by g to a $U(1)$ bundle over $P^{[2]}$. The fibre over (p_1, p_2) is the set of all $\hat{g}$ in $\hat{G}$ such that $p_1 \pi(\hat{g}) = p_2$. The product structure on $\hat{G}$ defines a bundle gerbe product. The resulting bundle gerbe is called the lifting bundle gerbe of $P \to X$.

Given the bundle $P \to X$ it is natural to ask if there is a $\hat{G}$ bundle $\hat{P} \to X$ such that $\hat{P}/U(1)$ is isomorphic to P as a G bundle. It is well known that this is true if and only if a certain class in $H^3(X, \mathbb{Z})$ vanishes. It is also easy to show [16] that such a lift is possible if and only if the lifting bundle gerbe is trivial. Moreover the class of the lifting bundle gerbe is the three class obstructing the lift.

The examples we need in this paper are torsion bundle gerbes. For these the central extension is of the form

$$\mathbb{Z}_d \to \hat{G} \xrightarrow{\pi} G \tag{4.2}$$

for some cyclic subgroup $\mathbb{Z}_d \subset U(1)$. In this case the obstruction to lifting the G bundle to a $\hat{G}$ bundle lives in $H^2(X, \mathbb{Z}_d)$ and again corresponds with the Dixmier–Douady class of the torsion bundle gerbe.

4.4. Stable isomorphism of bundle gerbes

For bundle gerbes there is a notion called *stable isomorphism* which corresponds exactly to two bundle gerbes having the same Dixmier–Douady class. To motivate this consider the case of two hermitian line bundles $L \to X$ and $J \to X$ they are isomorphic if there is a bijective map $L \to J$ preserving all structure, i.e. the projections to X and the $U(1)$ action on the fibres. Such isomorphisms are exactly the same thing as trivialisations of $L^* \otimes J$. For the case of bundle gerbes the latter is the correct notion and we have the following definition.

Definition 4.3. A stable isomorphism between bundle gerbes (L, Y) and (J, Z) is a trivialisation of $L^* \otimes J$.

We have from [17] the following proposition.

Proposition 4.4. *A stable isomorphism exists from (L, Y) to (J, Z) if and only if* $d(L) = d(J)$.

If a stable isomorphism exists from (L, Y) to (J, Z) we say that (L, Y) and (J, Z) are stably isomorphic.

It follows easily that stable isomorphism is an equivalence relation. It was shown in [16] that every class in $H^3(X, \mathbb{Z})$ is the Dixmier–Douady class of some bundle gerbe. Hence we can deduce from Proposition 4.4 the following theorem.

Theorem 4.5. *The Dixmier–Douady class defines a bijection between stable isomorphism classes of bundle gerbes and $H^3(X, \mathbb{Z})$.*

It is shown in [17] that a morphism from (L, Y) to (J, Z) induces a stable isomorphism but the converse is not true.

4.5. Bundle gerbe connections and curving

Let (L, Y) be a bundle gerbe over Y. Before defining connections we need a useful long exact sequence from [16]. Let $Y^{[p]} \to X$ be the pth fold fibre product of Y over the projection map to X. That is $Y^{[p]}$ is the subset of Y^p consisting of pairs $(y_1, \ldots, y_p)$ with the property that $\pi(y_1) = \pi(y_2) = \cdots = \pi(y_p)$. There are projection maps $\pi_i : Y^{[p]} \to Y^{[p-1]}$ which omit the ith component. We use these to define a map on differential forms

$$\delta : \Omega^q(Y^{[p-1]}) \to \Omega^q(Y^{[p]}) \tag{4.3}$$

by

$$\delta(\eta) = \sum_{i=1}^{p} (-1)^i \pi^*(\eta).$$

Note that δ commutes with exterior derivative. It is shown in [16] that the long sequence

$$0 \to \Omega^q(X) \xrightarrow{\pi^*} \Omega^q(Y) \xrightarrow{\delta} \Omega^q(Y^{[2]}) \xrightarrow{\delta} \Omega^q(Y^{[3]}) \xrightarrow{\delta} \cdots \tag{4.4}$$

is exact for every q.

A connection ∇ on $L \to Y^{[2]}$ is called a bundle gerbe connection if it commutes with the product structure on L. To be more precise, over $Y^{[3]}$, the bundle gerbe multiplication defines a bundle isomorphism $m : \pi_3^{-1}(L) \otimes \pi_1^{-1}(L) \to \pi_2^{-1}(L)$. On the bundle $\pi_3^{-1}(L) \otimes \pi_1^{-1}(L)$ we have the connection $\pi_3^{-1}(\nabla) \otimes \pi_1^{-1}(\nabla)$ and on $\pi_2^{-1}(L)$ the connection $\pi_2^{-1}(\nabla)$. We require that these are equal under the isomorphism m. It can be shown [16] that bundle gerbe connections exist. The curvature of a bundle gerbe connection F_∇ satisfies $\delta(F_\nabla) = 0$, where δ is defined in (4.3). Using the exactness of (4.4) we see that there is a (not unique) two-form f on Y satisfying $\delta(f) = F$. A choice of such an f we call a *curving* for the bundle gerbe connection. In string theory we would refer to f as the B-field. We have that $\delta(\mathrm{d}f) = \mathrm{d}\delta(f) = \mathrm{d}F = 0$ so, using exactness again, $\mathrm{d}f = \pi^*(\omega)$ for some three-form ω on X. As $\pi^*(\mathrm{d}\omega) = \mathrm{d}\pi^*(\omega) = \mathrm{d}\mathrm{d}f = 0$ we see that $\mathrm{d}\omega = 0$. The three-form ω is called the three-curvature of the bundle gerbe connection and curving. In string theory it is the H-field. As for line bundles the three-curvature represents the image, in real cohomology, of the Dixmier–Douady class.

We can extend the notion of stable isomorphism to bundle gerbes with connection and curving by saying that a bundle gerbe (L, Y) with connection ∇ and curving f is trivial if there is a line bundle $J \to Y$ with connection ∇_J and a bundle gerbe isomorphism $\delta(J) = L$ which maps $\delta(\nabla_J)$ to ∇ and for which $f = F_{\nabla_J}$. Then two bundle gerbes with connection and curving (L, Y) and (K, X) are stably isomorphic if $(L \otimes J^*, Y \times_f X)$ is trivial, as a bundle gerbe with connection and curving. Then we have the following theorem.

Theorem 4.6 ([17]). *The set of all stable isomorphism classes of bundle gerbes with connection and curving is equal to the Deligne cohomology* $H^3(X, \mathcal{D}^3)$.

4.6. Deligne cohomology of a bundle gerbe with connection and curving

An explicit map to Deligne cohomology can be defined as follows. Let $\{U_\alpha\}$ be a good open cover of X admitting local sections $s_\alpha : U_\alpha \to X$. We can define a map $s : Y_{\mathcal{U}} \to Y$, commuting with projections to X, by $s(\alpha, x) = s_\alpha(x)$. This induces maps $s^{[p]} : Y_{\mathcal{U}}^{[p]} \to Y^{[p]}$ which can be used to pull-back the line bundle $L \to Y^{[p]}$ to a line bundle $(s^{[2]})^{-1}(L) \to Y_{\mathcal{U}}^{[2]}$. As the pairwise intersections are contractible we can trivialise the line bundle by sections $\sigma_{\alpha\beta}$ over each $U_\alpha \cap U_\beta$. Then we can multiply $\sigma_{\alpha\beta}$ and $\sigma_{\beta\gamma}$ using the bundle gerbe product. Over $U_\alpha \cap U_\beta \cap U_\gamma$ we must have $\sigma_{\alpha\beta}\sigma_{\beta\gamma} = g_{\alpha\beta\gamma}\sigma_{\beta\gamma}$ for some function $g_{\alpha\beta\gamma}$ which is, in fact, a Cech cocycle. Also define $k_{\alpha\beta} \in \Omega^1(U_\alpha \cap U_\beta)$ by $\nabla\sigma_{\alpha\beta} = k_{\alpha\beta}\sigma_{\alpha\beta}$ and $f_\alpha \in \Omega^2(U_\alpha)$ by $f_\alpha = s_\alpha^*(f)$. In string theory this is how the B-field is usually presented as a collection of two-forms. The triple $[g_{\alpha\beta\gamma}, k_{\alpha\beta}, f_\alpha]$ defines a Deligne cohomology class. The curvature of this Deligne class is the three-curvature of the bundle gerbe connection and curving.

It follows that every bundle gerbe connection and curving defines a holonomy, that is an number in $U(1)$ assigned to any surface in X. To define this explicitly consider a bundle

gerbe with connection and curving over a surface Σ. Then as $H^3(\Sigma, \mathbb{Z}) = 0$ this is a trivial bundle gerbe with a trivialisation $J \to Y$. It can be shown [17] that we can find a connection ∇_J on J such that $\delta(\nabla_J) = \nabla$. We say such a connection is compatible with the bundle gerbe connection. Then $\delta(F_{\nabla_J} - f) = 0$ so that $F_{\nabla_J} - f = \mu_J$ for some two-form μ_J on Σ. Define

$$\text{hol}(\nabla, f, \Sigma) = \exp\left(\int_\Sigma \mu_J\right).$$

We leave it as an exercise to confirm that this is the same as the holonomy of the Deligne class constructed as in Section 3.2 from the bundle gerbe with connection and curving. As for the case of Deligne cohomology we often also compute holonomy of a map $\sigma : \Sigma \to X$ by first pulling the bundle gerbe with its connection and curving back to Σ.

As the bundle $L \to Y^{[2]}$ for a torsion bundle gerbe has a reduction to $\mathbb{Z}_d$ it has a canonical flat connection. Because the curvature of the flat connection vanishes the zero two-form on Y is a curving. The flat connection and zero curving provide a canonical choice of connection and curving for any torsion bundle gerbe. We leave it as an exercise for the reader to show that the Deligne cohomology class defined by the flat connection and zero curving is the canonical Deligne cohomology class of a class in $H^2(X, \mathbb{Z}_d)$ defined in Section 3.5.

4.7. Local bundle gerbes

If $\mathcal{U} = \{U_\alpha\}_{a \in I}$ is an open cover of X and we define $Y_\mathcal{U}$ as in (3.2) a bundle gerbe $(L, Y_\mathcal{U})$ is just a collection of line bundles $L_{\alpha\beta} \to U_\alpha \cap U_\beta$. This is a gerbe in the sense of Hitchin and Chatterjee. If we restrict further and require that the cover be good we can assume all the $L_{\alpha\beta}$ are trivial. In that case bundle gerbe multiplication must take the form

$$((\alpha, x), w) \otimes ((\beta, x), z) \mapsto ((\gamma, x), g_{\alpha\beta\gamma}(x)wz)$$

for some co-cycle $g_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cup U_\gamma \to U(1)$ and, moreover, a connection and curving define exactly a representative for a Deligne cohomology class in the double complex (3.4).

The local description of bundle gerbes follows from these results. Choose a good cover $\mathcal{U}$ and local sections $s_\alpha : U_\alpha \to Y$. Then these define a map $s : Y_\mathcal{U} \to Y$ by $s(\alpha, x) = s_a(x)$ which is fibre preserving. We can use this to pull-back the bundle gerbe (L, Y) to a stably isomorphic bundle gerbe $(s^{-1}(L), Y_\mathcal{U})$ and calculate locally.

4.8. Stable isomorphism and gauge transformations

In the case of abelian gauge theory we are interested in $U(1)$ bundles with connection and curving and these determine a Deligne one class. If we act on the bundle with a gauge transformation then the Deligne class is unchanged. The converse is also true. To see this let L be a bundle with connections A_1 and A_2 defining the same Deligne class. Pick a point $m_0 \in X$. For any other point m choose a path γ from m to m' and consider the parallel transports $P_1(\gamma)$ and $P_2(\gamma)$ from L_{m_0} to L_m. These define an isomorphism

$$P_2(\gamma)P_1(\gamma)^{-1} : L_m \to L_m.$$

If we choose another path γ' then we have

$$P_i(\gamma') = P_i(\gamma) \, \mathrm{hol}(\gamma \# \gamma', A_i)$$

but $\mathrm{hol}(\gamma \# \gamma', A_1) = \mathrm{hol}(\gamma \# \gamma', A_2)$ so that

$$P_2(\gamma) P_1(\gamma)^{-1} = P_2(\gamma') P_1(\gamma')^{-1},$$

and the result is a gauge transformation $g : L \to L$. Clearly this maps the parallel transport for A_1 to the parallel transport for A_2 and hence maps A_1 to A_2. We conclude that any two connections with the same Deligne class differ by a gauge transformation.

For bundle gerbes we know that any two bundle gerbes with the same Deligne class differ by a stable isomorphism with connection. Any two stable isomorphisms differ by a uniquely determined line bundle with connection in the sense that if $J \to Y$ and $K \to Y$ are stable isomorphisms then there is line bundle $L \to X$ such that $J = \pi^{-1}(L) \otimes K$. In addition L has a connection and the isomorphism $J = \pi^{-1}(L) \otimes K$ identifies the connection on J with the product of the pull-back connection on $\pi^{-1}(L)$ and the connection on K. Note that it is possible to compose stable isomorphisms but the composition is not associative [17,22].

In the case of stable isomorphisms from a bundle gerbe (P, Y) to itself the situation is somewhat simplified as we have a distinguished stable isomorphism—the identity. It follows that every stable isomorphism from (P, Y) to (P, Y) is determined by a line bundle J on X with connection ∇. We conclude that a gauge transformation of a bundle gerbe (L, Y) with connection ∇ and curving f is a line bundle $J \to X$ with connection D. Some calculation shows that it defines a stable isomorphism between (L, Y) with ∇ and f and (L, Y) with ∇ and $f + \pi^*(F_D)$, where F_D is the curvature of the connection D on $J \to X$. If we take local sections and represent the Deligne class of (L, Y) with ∇ and f by $(g_{\alpha\beta\gamma}, a_\alpha, f_{\alpha\beta})$ then the stable isomorphism changes it by addition of $D(k_{\alpha\beta}, A_\alpha) = (1, 0, \mathrm{d}A_\alpha)$, where $k_{\alpha\beta}$ are transition functions for J and A_α are local connection one-forms for D.

Note: Hitchin has remarked (Arbeitstagung lecture, Max Planck Institute, Bonn, 2001) that gauge transformations for gerbes form a category, they are certainly not a group.

4.9. Trivial bundle gerbes

Consider a bundle gerbe with connection and curving and Deligne class ξ. If the Dixmier–Douady class $(c(\xi))$ is zero then the bundle gerbe is trivial and we can repeat the discussion in the definition of holonomy in Section 4.6 and find a global trivialisation $J \to Y$ with connection ∇_J. The two-form μ_J is then a two-form on X. If we compare with the sequence (3.8) we can show that $\iota(\mu_J) = \xi$ the Deligne class of the bundle gerbe. As in Section 3.7 we can use μ_J to define a section χ_{μ_J} of L_ξ over $L(X)$.

If we change to another trivialisation J' and connection ∇' then there is a bundle $K \to X$ with connection ∇_K such that $J' = J \otimes \pi^{-1}(K)$, $\nabla_{J'} = \nabla_J \otimes \pi^{-1}(\nabla_K)$ and $\mu_{J'} = \mu_J + F_K$, where F_K is the curvature of ∇_K. Then we have (cf. (3.16))

$$\chi_{\mu_{J'}}(\sigma) = \mathrm{hol}(\nabla_K, \partial(\sigma)) \chi_{\mu_J}(\sigma).$$

Notice that the action of a gauge transformation is precisely that of tensoring the trivialisation J and its connection ∇_J with the pull-back of a line bundle $K \to X$ with connection ∇_K. It

follows that the change in the action (2.5) arising from changing the trivialisation χ_ρ can be regarded as resulting from a gauge transformation acting on the trivialisation. To be precise the gauge transformation (K, ∇_K) acting results in the value of the action on a world sheet σ being multiplied by

$$\exp\left(\int_\Sigma \sigma^*(F_K)\right),$$

where F_K is the curvature of ∇_K.

4.10. Bundle gerbe modules

Let (L, Y) be a bundle gerbe over a manifold X and let $E \to Y$ be a finite rank, hermitian vector bundle. Assume that there is a hermitian bundle isomorphism

$$\phi : L \otimes \pi_1^{-1}E \xrightarrow{\sim} \pi_2^{-1}E \tag{4.5}$$

which is compatible with the bundle gerbe multiplication in the sense that the two maps

$$L_{(y_1,y_2)} \otimes (L_{(y_2,y_3)} \otimes E_{y_3}) \to L_{(y_1,y_2)} \otimes E_{y_2} \to E_{y_1},$$

and

$$(L_{(y_1,y_2)} \otimes L_{(y_2,y_3)}) \otimes E_{y_3} \to L_{(y_1,y_3)} \otimes E_{y_3} \to E_{y_1}$$

are the same. In such a case we call E a bundle gerbe module and say that the bundle gerbe acts on E.

Notice that if E has rank one then it is a trivialisation of L. Moreover if E has rank r then L^r acts on $\wedge^r(E)$ and we deduce the following proposition.

Proposition 4.7. *If (L, Y) has a bundle gerbe module $Y \to E$ of rank r then its Dixmier–Douady class $\mathrm{d}(L)$ satisfies $r\mathrm{d}(L) = 0$.*

A connection ∇_E is called a bundle gerbe module connection if the bundle gerbe has a connection and the induced connections on $L \otimes \pi_1^{-1}E$ and $\pi_2^{-1}E$ are equal under the isomorphism (4.5).

If the bundle gerbe arises as the lifting bundle gerbe associated to a principal G bundle $P \to X$, where there is a central extension

$$U(1) \to \hat{G} \to G,$$

it follows from the definition of bundle gerbe module they are the same thing as bundles $E \to P$ with $\hat{G}$ action covering the G action on P and such that the action of $U(1)$ on any fibre E_p over $p \in P$ is scalar multiplication. For example in the case of

$$\mathbb{Z}_n \to SU(n) \to PU(n),$$

the trivial bundle $V \times P$ is a bundle gerbe module whenever V carries a representation of $SU(n)$.

4.11. Holonomy of bundle gerbe modules

We show in this subsection how, given a bundle gerbe module over a manifold, the holonomy of a connection on a bundle gerbe module defines a section of the line bundle (defined in Section 3.6) over the loop space of the manifold. Although the result is general our applications are when the manifold in question is the submanifold Q as in Section 2. Our construction is motivated by the construction in [12] using Azumaya algebra module connections.

Consider a bundle gerbe (R, Y) over Q with a connection and curving defining a torsion Deligne class ζ. The example we need in Section 2 is the lifting bundle gerbe for a $PU(n)$ principal bundle over Q. Let $E \to Y$ be a bundle gerbe module with a bundle gerbe module connection A. We wish to define a section $\operatorname{tr} \operatorname{hol}(A)$ of $L_\zeta \to L(Q)$ by constructing a function $s_A : D(Q) \to \mathbb{C}$ and showing that it transforms as in (3.12).

Let $\sigma : D \to Q$ be a map of a disk into Q and pull the bundle gerbe and connection and module back to D. Over D the bundle gerbe is trivial. Choose a trivialisation J with connection ∇_J compatible with the bundle gerbe connection and with curvature F_J. Then we have seen in Section 4.6 that $f - F_J = \pi^*(\mu_J)$ for some μ_J a two-form on D. Note also that $E \otimes J^*$ with connection $A - \nabla_J$ descends to a bundle E_J on D with connection D_J. We define

$$s_A : D(Q) \to \mathbb{C}$$

by

$$s_A(\sigma) = \operatorname{tr} \operatorname{hol}(D_J) \exp\left(\int_D \mu_J\right) \tag{4.6}$$

where the holonomy is computed over the boundary of σ. We need to check that s_A is independent of the choice of J and ∇_J.

Lemma 4.8. *The function* $s_A : D(Q) \to \mathbb{C}$ *depends only on A not on the choice of trivialisation J or connection* ∇_J.

Proof. If we change to another trivialisation J' with connection ∇'_J then there is line bundle K on D with connection ∇_K such that $J = \pi^{-1}(K) \otimes J'$ and $\nabla_J = \pi^{-1}(\nabla_K) \otimes \nabla'_J$. Similarly $E_J = E'_J \otimes K$ and $D_J = D'_J \otimes \nabla_K$. Hence

$$\operatorname{hol}(D_J) = \operatorname{hol}(D'_J \otimes \nabla_K) = \operatorname{hol}(D'_J) \operatorname{hol}(\nabla_K) = \operatorname{hol}(D'_J) \exp\left(-\int_D F_K\right)$$

so that

$$s_A(\sigma) = \operatorname{tr} \operatorname{hol}(D_J) \exp\left(\int_D \mu_J\right) = \operatorname{tr} \operatorname{hol}(D'_J) \exp\left(-\int_D F_K\right) \exp\left(\int_D \mu_J\right)$$

$$= \operatorname{tr} \operatorname{hol}(D'_J) \exp\left(\int_D \mu'_J\right),$$

and the function s_A is independent of J and ∇_J.   □

Next we have the following lemma.

Lemma 4.9. *The function s_A transforms as a section of L_ζ.*

Proof. Assume now that we have two maps $\sigma_i : D \to Q$ which agree with σ on the boundary. Trivialise the pull-back by $\sigma_1 \# \sigma_2$ of L_ζ over the whole of the two-sphere. Denote this trivialisation by J and its connection by ∇_J and use subscripts i to denote the restrictions to the two hemispheres D_1 and D_2. Then

$$s(\sigma_1) = \operatorname{tr} \operatorname{hol}(D_{J_1}) \exp\left(\int_{D_1} \mu_{J_1} \right) = \operatorname{tr} \operatorname{hol}(D_{J_2}) \exp\left(\int_{D_1} \mu_{J_1} \right)$$

$$= \operatorname{tr} \operatorname{hol}(D_{J_2}) \exp\left(\int_{D_2} \mu_{J_2} \exp\left(\int_{S^2} \mu_J \right) \right) = s(\sigma_2) \exp\left(\int_{S^2} \mu_J \right),$$

so that s is a section of L_ξ. We use here the fact that σ_1 and σ_2 agree on the boundary of D and that D_{J_1} and D_{J_2} agree on this common boundary. $\qquad\square$

We now define the section $\operatorname{tr} \operatorname{hol}(A) : L(Q) \to L_\zeta$ to be that given by the function s_A. This means we have defined all of the terms in the tensor product (2.7). That the result is a function is a consequence of these definitions.

In the case that the bundle gerbe is not torsion it was shown in [2] that twisted K-theory could be constructed from bundle gerbe modules $E \to Y$ whose structure group was reduced to the group of unitaries on an infinite dimensional Hilbert space $\mathcal{H}$ (isomorphic to the fibres of E) which differ from the identity by a compact operator. If we require a slightly stronger result, that the bundle gerbe module have a reduction to the group of unitaries on $\mathcal{H}$ that differ from the identity by something which is trace class then in the formula (4.6) the quantity $\operatorname{hol}(D_J)$ is a unitary differing from the identity by a trace class operator. Choose now two bundle gerbe module connections A_1 and A_2 on E so we have $\operatorname{hol}(D_{1,J})$ and $\operatorname{hol}(D_{2,J})$ which are unitaries differing from the identity by a trace class operator. Hence we can define

$$s(\sigma) = \operatorname{tr}(\operatorname{hol}(D_{1,J}) - \operatorname{hol}(D_{1,2})) \exp\left(\int_D \mu_J \right).$$

To see that this is well-defined and a section of L_ζ is a repeat of the calculation above. We have

$$\operatorname{hol}(D_{i,J}) = \operatorname{hol}(D_{i,J'}) \operatorname{hol}(\nabla_K)$$

for $i = 1, 2$ so that

$$\operatorname{hol}(D_{1,J}) - \operatorname{hol}(D_{2,J}) = (\operatorname{hol}(D_{1,J'}) - \operatorname{hol}(D_{2,J'})) \operatorname{hol}(\nabla_K),$$

giving

$$\operatorname{tr}(\operatorname{hol}(D_{1,J}) - \operatorname{hol}(D_{2,J})) = \operatorname{tr}(\operatorname{hol}(D_{1,J'}) - \operatorname{hol}(D_{2,J'})) \operatorname{hol}(\nabla_K),$$

and the argument goes through as above to define a section $\operatorname{tr}(\operatorname{hol}(A_1) - \operatorname{hol}(A_2))$ of L_ζ over $L(Q)$.

We have made sense of all of the terms in the tensor product (2.7) and by construction it is a well-defined function on $\Sigma_Q(M)$.

5. Torsion bundle gerbes and C^*-algebras

In this section we show how to relate torsion bundle gerbes to certain continuous trace C^*-algebras. In the course of this discussion we will explain the relation between Kapustin's work [2,12].

We start with a principal bundle $\pi : Y \to X$ with structure group $PU(n)$. All torsion elements of $H^3(X, \mathbb{Z})$ (Cech cohomology) arise as the Dixmier–Douady class of the lifting bundle gerbe $P \to Y^{[2]}$ associated to $\pi : Y \to X$ for some choice of n. Fix one torsion class in $H^3(X, Z)$ and let P be the lifting bundle gerbe associated to this class by the central extension

$$\mathbb{Z}_n \to U(n) \to PU(n).$$

We use the theory of locally compact groupoid C^*-algebras as developed by [14,15,20]. To this end observe that $Y^{[2]}$ is the groupoid of a relation on Y namely we say $y_1 \sim y_2$ if y_1 and y_2 lie in the same fibre of $\pi : Y \to X$. The set of equivalence classes under this relation is X. In fact $Y^{[2]}$ is a proper groupoid with unit space Y because it is easy to check that it satisfies the requirement [14] that the map $\pi_0 : Y^{[2]} \to Y \times Y$ which regards $Y^{[2]}$ as a subset of the product $Y \times Y$ is a homeomorphism onto a closed subset of the product space. Note that the maps π_1 and π_2 from $Y^{[2]} \to Y$ are the range and source maps respectively of this groupoid which has, as its operations, the product $(y_1, y_2)(y_2, y_3) = (y_1, y_3)$ and inverse $(y_1, y_2)^{-1} = (y_2, y_1)$. We identify the unit space Y with the diagonal $\{(y, y)|y \in Y\}$.

Now we remark that $Y^{[2]}$ is locally compact and admits a Haar system. We recall construction of the latter. As $Y \to X$ admits local sections we can use the resulting local trivialisation to choose for $(y_1, y_2) \in Y^{[2]}$ a measure λ^{y_1} on the $\{(y_1, y)|y \in Y, \pi(y) = \pi(y_1)\} \subset Y^{[2]}$. In fact we may take for λ^{y_1}, Haar measure on $PU(n)$ as the measure on $\{(y_1, y)|y \in Y, \pi(y) = \pi(y_1)\}$ using the local trivialisation to identify these spaces. Note that a set $\{(y_1, y)|y \in Y, \pi(y) = \pi(y_1)\}$ may be identified with $PU(n)$ in many ways depending on which open set of the cover we choose. However, we fix one choice for each fibre throughout. This involves a choice from only finitely many options as our space X is paracompact and the cover of X is locally finite. The set of measures $\{\lambda^{y_1}|y_1 \in Y\}$ is easily seen to define a Haar system on $Y^{[2]}$. We remark that there is one technical condition on a Haar system that may not be obvious. This is that if $C_c(Y^{[2]})$ denotes the continuous functions of compact support on $Y[2]$ then we have for all $f \in C_c(Y^{[2]})$ that the map $(y_1, y_2) \to \int f(y_1, y) \, d\lambda^{y_1}(y_1, y)$ is continuous. After a moments thought one sees that the construction of our measures via the local trivialisation guarantees this.

We may describe the groupoid structure on P in a number of ways. To make use of the results of [15] we will use the language of principal $BbbT$-groupoids. This means that we will regard P as an extension of the groupoid $Y^{[2]}$ in the sense of Definition 2.2 of [13]. To this end we observe that $P/\mathbb{T} \equiv Y^{[2]}$ because P is a $U(1)$ bundle over $Y^{[2]}$. We may define the range and source maps of P to be $r, s : P_{(y_1, y_2)} \to Y$, where $r(z) = (y_1, y_1)$ and $s(z) = (y_2, y_2)$ for $z \in P_{(y_1, y_2)}$. The sense in which P is an extension of $Y^{[2]}$ arises from the

existence of a two-cocycle on $Y^{[2]}$ defined via the multiplication in P. Recall that $P_{(y_1, y_2)}$ consists of those elements u of $U(n)$ such that $y_1.p(u) = y_2$, where $p : U(n) \to PU(n)$ is the projection. We will regard our extension of $PU(n)$ as a set of pairs (g, t), where $g \in PU(n)$ and $t \in \mathbb{Z}_n$. This can be achieved globally by choosing a Borel cross-section c of p. Note that as p has discrete fibres we may choose c to be locally constant. The multiplication in $U(n)$ is then written

$$(g_1, c(g_1))(g_2, c(g_2)) = (g_1 g_2, c(g_1 g_2)\omega(g_1, g_2)) \qquad (*)$$

where ω is a group two-cocycle on $PU(n)$. It is now not hard to recognise P as a principal $\mathbb{T}$-groupoid as described in [14].

The next step is to identify the Dixmier–Douady class of P regarded as a bundle gerbe. It is determined by choosing a good cover $\{U_\alpha\}$ of X and transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to Y$ for the bundle $Y \to X$. Then the Dixmier–Douady class of P is defined by the multiplication on P. We can write this multiplication using the locally constant cross-section c and $(*)$ as

$$c(g_{\alpha\beta}(m))c(g_{\beta\gamma}(m)) = c(g_{\alpha\gamma})\omega(g_{\alpha\beta}(m), g_{\beta\gamma}(m)).$$

It follows from this that ω determines the Dixmier–Douady class of P as a bundle gerbe.

Now we need to describe the C^*-algebra associated with this principal $\mathbb{T}$-groupoid P. Let $\Gamma_c(P)$ denote the sections of $P \to Y^{[2]}$ which are of compact support. These may be thought of as functions: $f : P \to \mathbb{C}$ satisfying $f(z.t) = tf(z)$ for $z \in P$. There is a multiplication on $\Gamma_c(P)$ given by

$$f * g(z_1) = \int f(z_1 z_2) g(z_2^{-1}) \mathrm{d}\lambda^{s(z_1)}(\dot{z}_2),$$

where $\dot{z}_2$ is the image of z_2 under $P \to Y^{[2]}$. The involution is

$$f^*(z_1) = \bar{f}(z_1^{-1}).$$

We denote by $C^*(P, Y^{[2]}, \lambda)$ the C^* completion of $\Gamma_c(P)$ following the notation and definitions of [14].

The conclusion of the main result of [14] is that the principal $\mathbb{T}$-groupoid C^*-algebra $C^*(P, Y^{[2]}, \lambda)$ is continuous trace with spectrum X. The technical assumption of [15] that $Y \to X$ admits local sections is clearly satisfied so that we may apply Section 5 of [15]. This states that the Dixmier–Douady class of $C^*(P, Y^{[2]}, \lambda)$, is the obstruction to $C^*(P, Y^{[2]}, \lambda)$ being Morita equivalent to the C^*-algebra of continuous functions on X which vanish at infinity $C_0(X)$.

We need to verify that the Dixmier–Douady class of $C^*(P, Y^{[2]}, \lambda)$ is the same as the Dixmier–Douady class of P as a bundle gerbe. This is notationally messy and to save space we refer the reader to 128 pp. of [15]. There, in the discussion centring around equations (5.5) and (5.6), the Dixmier–Douady class of $C^*(P, Y^{[2]}, \lambda)$ is shown to arise from ω in essentially the same fashion as does the Dixmier–Douady class of P as a bundle gerbe.

Using the known fact [21] that two continuous trace C^*-algebras with the same spectrum and Dixmier–Douady class are Morita equivalent we conclude that if we have an Azumaya

algebra over X as in [12] with the same Dixmier–Douady class as P then it must be Morita equivalent to $C^*(P, Y^{[2]}, \lambda)$.

We can see that the bundle gerbe module E for P is a module for $C^*(P, Y^{[2]}, \lambda)$. Let $f \in \Gamma_c(P)$ and let ξ be a smooth, compactly supported section of E. It is convenient at this point to regard the bundle gerbe action on E as being given by a map $\phi(y_1, y_2) : E_{y_2} \to E_{y_1}$. With this choice we can integrate the P action up a fibre. So if ξ is a section of E we write

$$\tilde{\phi}(y_1, y_2)\xi(y_1, y_1) = \phi(y_1, y_2)\xi[(y_2, y_1)(y_1, y_1)(y_1, y_2)] = \phi(y_1, y_2)\xi(y_2, y_2).$$

The reason for the conjugation action of the groupoid $Y^{[2]}$ on Y is that this is how it acts on the diagonal. Now we have to integrate this to get an action of $f \in \Gamma_c(P)$. We set, for $z \in P_{y_1, y_2}, \dot{z} = (y_1, y_2)$ and define

$$f\xi(r(w)) := \int f(z)\phi(\dot{z})\xi(s(w)) \, d\lambda^{r(w)}(\dot{z}).$$

We could go further but desist at this point for the reason that we do not know how to make the above discussion work when the Dixmier–Douady class is non-torsion. Indeed this was the whole reason for introducing bundle gerbe modules in the first place: they give a realisation of twisted K^0 without the need to introduce operator algebras.

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